

Quantum filtering: a reference probability approach

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Abstract

These notes are intended as an introduction to noncommutative (quantum) filtering theory. An introduction to quantum probability theory is given, focusing on the spectral theorem and the conditional expectation as the least squares estimate, and culminating in the construction of Wiener and Poisson processes on the Fock space. Next we describe the Hudson-Parthasarathy quantum Itô calculus and its use in the modelling of physical systems. Finally, we use a reference probability method to obtain quantum filtering equations, in the Belavkin-Zakai (unnormalized) form, for several system-observation models from quantum optics. The normalized (Belavkin-Kushner-Stratonovich) form is obtained through a noncommutative analogue of the Kallianpur-Striebel formula.

1 Introduction

The development of a theory of feedback control in the 1950s and 60s has provided a huge stimulus to the engineering disciplines. Nowadays all but the simplest of devices that make up our everyday lives use feedback to provide efficient and reliable performance despite the ever increasing complexity and miniaturization. However, at a time when microtechnology is making way for nanotechnology, we are rapidly approaching the boundary of the classical world past which the effects of quantum mechanics cannot be neglected. The theory of quantum mechanics tells us that any description of the phenomena at small scales is inherently nondeterministic in nature. This opens new areas of application for stochastic control theory, which will likely play an important role in a future generation of technology. In particular, as observations of quantum systems are inherently noisy, the theory of filtering—the extraction of information from a noisy signal—forms an essential part of any quantum feedback control strategy.

These notes are intended as an introduction to quantum filtering theory. We begin by giving a brief overview of noncommutative probability theory [45, 15], the associated stochastic calculus [35], and the role of this theory in the modelling of quantum dynamical systems [2, 31, 32]. We then obtain quantum filtering equations [13] for a variety of quantum models. We systematically

use change of measure techniques, bringing our approach close to the classical reference probability method of Zakai [56].

The study of quantum filtering and control was pioneered by V.P. Belavkin in a remarkable series of articles, dating back to the early 1980s [9, 10, 11, 12, 13, 14]. A discrete version of his filtering equation can already be found in [9]. After the development of quantum stochastic calculus [35], continuous time versions of Belavkin's equation appeared in [11], [12] and [13]. The sample paths of the solution of the quantum filtering equation [21, 22, 19, 55] are known in the continuous time quantum measurement community as quantum trajectories.

The reference probability method used here is inspired by the classical treatment of Zakai [56]. In this approach we change the underlying measure so that the filtering problem reduces to elementary manipulations of the conditional expectation with respect to the new (reference) measure. The key idea that allows us to apply this approach to the quantum case is the observation that the Radon-Nikodym derivative must be compatible with the observations that we are conditioning on. To obtain a change of measure that satisfies this property we employ a technique that to our knowledge first appeared in a paper by Holevo [34] (see also [12, 29, 8]), which replaces here the Girsanov transformation in Zakai's treatment. Apart from this point, the derivation is essentially classical, i.e. we derive the Belavkin-Zakai equation and obtain the (nonlinear) Belavkin-Kushner-Stratonovich equation via a noncommutative analogue of the Kallianpur-Striebel formula.

In contrast to the martingale techniques used in [13, 18], the reference probability method is independent from the innovations conjecture [23] (as is the case in classical filtering). Moreover, the reference probability approach consists only of elementary manipulations of the (quantum) conditional expectation. As in classical filtering theory the Kallianpur-Striebel formula has found a wide range of applications, it is our hope that a similar approach will be fruitful in the quantum context and will clarify some of the existing literature on quantum filtering theory.

After the filtering equations have been obtained, methods from classical nonlinear stochastic control can be applied [10, 52, 17] to design control laws. Thus the presence of quantum mechanics in quantum feedback control remains limited to the filtering procedure. Recent experiments implementing quantum feedback controls [4, 27, 28] have led to renewed interest in the field which is now rapidly expanding [24, 52, 17, 36, 37, 33, 25, 51, 53, 18]. The interaction between the areas of stochastic control and theoretical and experimental physics is essential in paving the way towards the engineering of quantum technologies.

There are three key ingredients that are required for the development of quantum filtering theory. First, we need to capture both classical probability and quantum mechanics within the framework of a generalised probability theory, called noncommutative or quantum probability theory. The central object in this theory, the spectral theorem, allows us to make a seamless connection between quantum systems and the associated classical probabilistic measurement outcomes. Second, we need a noncommutative generalization of the concept of conditional expectations. As in classical probability, we will find that a suitably restricted definition of the quantum conditional expectation is none other than a least squares estimator, which elucidates its role in quantum filtering theory. Finally, we need a noncommutative analogue of a stochastic calculus and of stochastic differential equations. This construction provides a broad class of models for which we can obtain quantum filtering equations.

In the following we develop each of these themes in turn. Section 2 introduces noncommutative probability theory, focusing on the spectral theorem, and introduces the quantum conditional expectation as a least squares estimate. In section 3 we show how Wiener and Poisson processes emerge in a particular quantum probabilistic model based on the Fock space. The construction additionally demonstrates the use of the spectral theorem. Section 4 is an exposition of the noncommutative stochastic calculus of Hudson and Parthasarathy. In Section 5 we introduce the system-observation models which we are interested in; they are obtained from physical models via a weak coupling limit. Section 6 deals with the derivation of the Belavkin-Zakai and the Belavkin-Kushner-Stratonovich equation using the reference probability approach. We conclude these notes with some additional examples.

2 Noncommutative probability theory

In this section we wish to make the point that quantum mechanics is a probability theory in which the random variables, called observables in quantum mechanics, are allowed to be noncommutative. Indeed, in quantum theory observables are represented by self-adjoint operators on some Hilbert space \mathcal{H} which in general need not commute. In order to incorporate quantum mechanics in the framework of probability theory we will weaken the axioms of the latter to allow for noncommutative random variables. By considering commuting observables only, classical probability theory can be recovered using the spectral theorem. The latter asserts that commuting self-adjoint operators can be diagonalised simultaneously, allowing them to be interpreted simultaneously as functions (i.e. random variables) on the diagonal (the spectrum). Excellent introductions to noncommutative probability theory are [45, 15], see also [16, 52].

As quantum probability is algebraic in nature, it is instructive to begin by forming an algebraic picture of classical probability theory. Consider a classical probability space $(\Omega, \Sigma, \mathbf{P})$. Then the space $\mathcal{A} = L^\infty(\Omega, \Sigma, \mathbf{P})$ of bounded measurable complex functions on Ω (we will always use *complex* function spaces L^∞ , L^2 etc.) is a $*$ -algebra: any complex linear combination, pointwise multiplication, and conjugate of functions in \mathcal{A} are still in \mathcal{A} . We can use integration with respect to \mathbf{P} to map any element of \mathcal{A} to a complex number. If we operate this map $\mathbb{P} : \mathcal{A} \rightarrow \mathbb{C}$ on an indicator function, we obtain the probability of the corresponding event. As the algebra only contains measurable functions, and as it is in fact generated by the indicator functions (any element in L^∞ can be approximated arbitrarily well by linear combinations of its indicator functions), the pair $(\mathcal{A}, \mathbb{P})$ encodes the same information as the probability space $(\Omega, \Sigma, \mathbf{P})$ (at least up to sets of measure zero [45], a point on which we will not dwell further.) It is convenient to represent the elements of the algebra $\mathcal{A} = L^\infty(\Omega, \Sigma, \mathbf{P})$ as *operators* on the Hilbert space $\mathcal{H} = L^2(\Omega, \Sigma, \mathbf{P})$. This can be done by letting \mathcal{A} act on \mathcal{H} via pointwise multiplication (note that $A\psi$ is an element of \mathcal{H} for any $A \in \mathcal{A}$, $\psi \in \mathcal{H}$).

Let us now generalize these ideas. Denote by $\mathcal{B}(\mathcal{H})$ the algebra of all bounded operators on a Hilbert space \mathcal{H} , and let $\mathcal{S} \subset \mathcal{B}(\mathcal{H})$. The set $\mathcal{S}' := \{X \in \mathcal{B}(\mathcal{H}); XS = SX, \forall S \in \mathcal{S}\}$ is called the *commutant* of \mathcal{S} in $\mathcal{B}(\mathcal{H})$. A *von Neumann algebra* \mathcal{A} on \mathcal{H} is a $*$ -subalgebra of $\mathcal{B}(\mathcal{H})$ such that \mathcal{A} equals its double commutant, i.e. $\mathcal{A}'' = \mathcal{A}$. Von Neumann's double commutant theorem then asserts that \mathcal{A} is closed in the weak operator topology, i.e. if for some net $\{A_j\} \in \mathcal{A}$ the inner products $\langle x, A_j y \rangle \rightarrow \langle x, Ay \rangle$ for all $x, y \in \mathcal{H}$, then A is an element of \mathcal{A} . This property guarantees that a von Neumann algebra is generated by its projections $P \in \mathcal{A}$, $P = P^2 = P^*$ [38].

From $\mathcal{A}'' = \mathcal{A}$ it immediately follows that the identity I is an element of the von Neumann algebra \mathcal{A} . A *state* on \mathcal{A} is a linear map $\mathbb{P} : \mathcal{A} \rightarrow \mathbb{C}$ such that \mathbb{P} is *positive* in the sense that $\mathbb{P}(A^*A) \geq 0$ for all $A \in \mathcal{A}$ and such that \mathbb{P} is normalised, i.e. $\mathbb{P}(I) = 1$. A state is called *normal* if it is weak operator continuous on the unit ball of \mathcal{A} . It is called *faithful* if $\mathbb{P}(A^*A) = 0$ implies $A = 0$. We remark that $L^\infty(\Omega, \Sigma, \mathbf{P})$ is a von Neumann algebra on $L^2(\Omega, \Sigma, \mathbf{P})$ with normal state \mathbb{P} .

The following theorem (see [39] sections 9.3–9.5 for a proof) is at the heart of spectral theory.

Theorem 2.1: (Spectral Theorem) *Let \mathcal{C} be a commutative von Neumann algebra and \mathbb{P} a normal state on \mathcal{C} . Then there is a probability space $(\Omega, \Sigma, \mathbf{P})$ such that \mathcal{C} is $*$ -isomorphic to $L^\infty(\Omega, \Sigma, \mathbf{P})$, the space of all bounded measurable functions on Ω . Furthermore, if we denote the $*$ -isomorphism by $\iota : \mathcal{C} \rightarrow L^\infty(\Omega, \Sigma, \mathbf{P})$ then we have*

$$\mathbb{P}(C) = \int_{\Omega} \iota(C) \mathbf{P}(d\omega), \quad C \in \mathcal{C}.$$

Example. We are guided by the elementary case of a Hilbert space with dimension $n < \infty$. Any linear operator on such a space can be represented as a (complex) $n \times n$ matrix, and any $*$ -algebra of matrices is a von Neumann algebra. Now consider a set of matrices $\{A_i\}$ that commute with each other and with all their adjoints, $[A_i, A_j] = [A_i, A_j^*] = 0 \ \forall i, j$. These matrices generate a commutative von Neumann (matrix) algebra \mathcal{C} . We can now simultaneously diagonalize every element of \mathcal{C} using a unitary U . To each $A \in \mathcal{C}$, we associate a map $\iota(A) : \{1, \dots, n\} \rightarrow \mathbb{C}$ such that $\iota(A)(i)$ is the i th diagonal element of the diagonalized matrix U^*AU . Then $\iota(A) \in L^\infty(\{1, \dots, n\})$, and $\iota(AB) = \iota(A)\iota(B)$ under pointwise multiplication in L^∞ . Now choose $\Sigma = \sigma(\iota(C); C \in \mathcal{C})$, the sigma-algebra generated by $\iota(\mathcal{C})$. Finally, given a state \mathbb{P} on \mathcal{C} , define for $\Gamma \in \Sigma$ the measure $\mathbf{P}(\Gamma) = \mathbb{P}(UC^\Gamma U^*)$ where $C_{ii}^\Gamma = \chi_\Gamma(i)$ (χ_Γ is the indicator function of Γ). Then we have explicitly constructed a $*$ -isomorphism $\iota : \mathcal{C} \rightarrow L^\infty(\{1, \dots, n\}, \Sigma, \mathbf{P})$. The spectral theorem is an extension of these ideas to infinite dimensional operators. Though technically much more involved, the flavour of the procedure remains the same.

We have already described how a classical probability space can be encoded algebraically by a commutative von Neumann algebra with normal state. The spectral theorem allows us to conclude that studying commutative von Neumann algebras equipped with normal states is equivalent to studying probability spaces. This motivates the definition of a *noncommutative* or *quantum* probability space as a von Neumann algebra \mathcal{N} equipped with a normal state \mathbb{P} . The events in this theory are the projections in \mathcal{N} , and the state plays the role of a probability measure. We can now see how the various technical properties of von Neumann algebras contribute to their interpretation as probabilistic models. Weak closure guarantees that the theory is completely determined by the set of all events; and normality of the state is equivalent to countable additivity [39].

Any physical experiment can be described by classical probability, provided that we consistently perform the same measurements. For example, if we measure the position of a particle at time t , and repeat this experiment many times with the same initial conditions, then the statistics of the measurement outcomes are entirely described by a classical probabilistic model. The break with classical probability occurs because in quantum mechanics there exist observables, such as position and momentum, that cannot be measured simultaneously in the same experimental realization. Such measurements are said to be *incompatible*, and this is enforced by the fact that they do not commute. The spectral theorem provides a concrete mathematical implementation of these ideas:

a set of compatible events generates a commutative algebra, and is hence equivalent to a classical probabilistic model. Thus if E and F are compatible events we can interpret EF as the event $\langle E \text{ and } F \rangle$ and $E \vee F := E + F - EF$ as the event $\langle E \text{ or } F \rangle$. However, two incompatible events are not represented on the same classical probability space, and their joint probability is undefined.

Having considered the noncommutative analogue of a probability space, let us now turn our attention to random variables. Let us first investigate a classical random variable $f : \Omega \rightarrow \mathbb{R}$. If f is bounded, we could interpret it as an element of the algebra $\mathcal{N} = L^\infty(\Omega, \Sigma, \mathbf{P})$. In general, however, a random variable need not be bounded. Nonetheless we can still consider f as a (self-adjoint) operator acting on $\mathcal{H} = L^2(\Omega, \Sigma, \mathbf{P})$ by pointwise multiplication, provided that we restrict its domain to the set of $\psi \in \mathcal{H}$ such that $f\psi \in \mathcal{H}$. To make the algebraic description self-contained, we must now express the fact that f is measurable in terms of the algebra \mathcal{N} . This can be done by requiring that for every Borel set $B \subset \mathbb{R}$ the indicator function $\chi_{\{\omega \in \Omega : f(\omega) \in B\}}$, considered as an operator on \mathcal{H} by pointwise multiplication, is an element of \mathcal{N} .

A random variable on a quantum probability space, called an *observable*, is a self-adjoint operator on \mathcal{H} . If an observable F is an element of \mathcal{N} (and hence is bounded), then it generates a commutative von Neumann algebra and we can use the spectral theorem to map it to a random variable $\iota(F)$ on a classical probability space. In general, however, F need not be bounded; we must extend the map ι to be able to represent unbounded observables as classical random variables. To this end, note that since F is self-adjoint its spectrum is real. Therefore we can define two bounded commuting operators $T_+ := (F + iI)^{-1}$ and $T_- := (F - iI)^{-1}$. The operators T_+ and T_- generate a commutative von Neumann algebra, and can hence be represented on a classical probability space by the spectral theorem. We now extend ι as

$$\iota(F) := \frac{(\iota(T_+))^{-1} + (\iota(T_-))^{-1}}{2}.$$

Then $\iota(F)$ is the representation of F as a classical random variable. Now define the *spectral measure*

$$E(B) = \iota^{-1}(\chi_{\{\omega \in \Omega : \iota(F)(\omega) \in B\}})$$

for any Borel set B of \mathbb{R} . $E(B)$ is the event in \mathcal{N} corresponding to $\langle F \text{ takes a value in } B \rangle$. F is *affiliated* to a von Neumann algebra \mathcal{C} if for all Borel sets B we have $E(B) \in \mathcal{C}$. This concept is equivalent to measurability of a classical random variable. Finally, note that if the observable F is an element of the algebra, then the expectation of F is given by $\mathbb{P}(F)$. This is consistent with the interpretation of F as a classical random variable $\iota(F)$, as $\mathbb{P}(F) = \mathbb{E}_{\mathbf{P}}[\iota(F)]$. Using the extension of ι we can extend the state \mathbb{P} to unbounded observables by

$$\mathbb{P}(F) = \mathbb{E}_{\mathbf{P}}[\iota(F)] = \int_{\mathbb{R}} \lambda \mathbb{P}(E(d\lambda))$$

Then $\mathbb{P}(F)$ is the expectation of the unbounded observable F .

Definition 2.2: (Conditional expectation) Let $(\mathcal{A}, \mathbb{P})$ be a quantum probability space and let $\mathcal{C} \subset \mathcal{Z} = \{Z \in \mathcal{A}; AZ = ZA, \forall A \in \mathcal{A}\}$, the *center* of \mathcal{A} . Then $\mathbb{P}(\cdot|\mathcal{C}) : \mathcal{A} \rightarrow \mathcal{C}$ is (a version of) the *conditional expectation* from \mathcal{A} onto \mathcal{C} if for all $A \in \mathcal{A}$, we have $\mathbb{P}(\mathbb{P}(A|\mathcal{C})C) = \mathbb{P}(AC) \forall C \in \mathcal{C}$.

The center \mathcal{Z} is by definition a commutative algebra. In applications, we begin with a fixed probability space $(\mathcal{N}, \mathbb{P})$ and specify a commutative von Neumann subalgebra $\mathcal{Z} \subset \mathcal{N}$ generated

by the observations we have performed. We then choose $\mathcal{A} = \mathcal{Z}'$, the commutant of \mathcal{Z} in \mathcal{N} . By the double commutant theorem, \mathcal{Z} is the center of \mathcal{Z}' . Def. 2.2 only allows conditioning of operators that commute with all the elements of the commutative algebra \mathcal{Z} . This is a natural requirement, as there is no need for updating observables that are incompatible with what has already been observed. Note that $\iota(\mathbb{P}(A|\mathcal{C})) = \mathbb{E}_{\mathbf{P}}[\iota(A)|\sigma(\iota(C); C \in \mathcal{C})]$ for any self-adjoint $A \in \mathcal{A}$, where ι is given by the spectral theorem applied to the commutative von Neumann algebra generated by A and \mathcal{C} . Hence for observables the conditional expectation coincides with its classical counterpart.

Remark. We have taken a more restrictive definition of the conditional expectation than is usual in quantum probability, i.e. we do not allow for conditioning on noncommutative algebras. The more general definition [50] does not have a direct physical interpretation but is required for the definition of concepts such as a noncommutative Markov process [40]. Beside its direct physical meaning, one advantage of our more restrictive approach is that a conditional expectation from \mathcal{A} onto \mathcal{C} in the above sense always exists (see [18] for a “construction” using the central decomposition of \mathcal{A}) and is unique with probability one. In fact, our conditional expectation is a special case of the conditional expectation of [50].

Lemma 2.3: *The conditional expectation of Def. 2.2 exists and is unique with probability one (i.e., any two versions P and Q of $\mathbb{P}(A|\mathcal{C})$ satisfy $\|P - Q\|_{\mathbb{P}} = 0$, where $\|X\|_{\mathbb{P}}^2 := \mathbb{P}(X^*X)$.) Moreover, $\mathbb{P}(A|\mathcal{C})$ is the least mean square estimate of A given \mathcal{C} , i.e. $\|A - \mathbb{P}(A|\mathcal{C})\|_{\mathbb{P}} \leq \|A - C\|_{\mathbb{P}} \forall C \in \mathcal{C}$.*

Proof. 1. *Existence.* We have already established that for self-adjoint $A \in \mathcal{A}$, we can explicitly define a $\mathbb{P}(A|\mathcal{C})$ that satisfies the conditions of Def. 2.2 using the spectral theorem, i.e. $\mathbb{P}(A|\mathcal{C}) = \iota^{-1}(\mathbb{E}_{\mathbf{P}}[\iota(A)|\sigma(\iota(C); C \in \mathcal{C})])$. The classical conditional expectation exists, and moreover the conditional expectation of a bounded random variable is bounded. Hence $\mathbb{P}(A|\mathcal{C})$ exists in \mathcal{C} for self-adjoint $A \in \mathcal{A}$. But any $A \in \mathcal{A}$ can be written as $A = A_1 + iA_2$ with self-adjoint $A_1 = (A + A^*)/2$ and $A_2 = i(A^* - A)/2$. As $\mathbb{P}(A_1|\mathcal{C})$ and $\mathbb{P}(A_2|\mathcal{C})$ exist and $\mathbb{P}(A|\mathcal{C}) = \mathbb{P}(A_1|\mathcal{C}) + i\mathbb{P}(A_2|\mathcal{C})$ satisfies the conditions of Def. 2.2, existence is proved.

2. *Uniqueness w.p. one.* Define the pre-inner product $\langle X, Y \rangle := \mathbb{P}(X^*Y)$ on \mathcal{A} (it might have nontrivial kernel if \mathbb{P} is not faithful.) Then $\langle C, A - \mathbb{P}(A|\mathcal{C}) \rangle = \mathbb{P}(C^*A) - \mathbb{P}(C^*\mathbb{P}(A|\mathcal{C})) = 0$ for all $C \in \mathcal{C}$ and $A \in \mathcal{A}$, i.e. $A - \mathbb{P}(A|\mathcal{C})$ is orthogonal to \mathcal{C} . Now let P and Q be two versions of $\mathbb{P}(A|\mathcal{C})$. It follows that $\langle C, P - Q \rangle = 0$ for all $C \in \mathcal{C}$. But $P - Q \in \mathcal{C}$, so $\langle P - Q, P - Q \rangle = \|P - Q\|_{\mathbb{P}}^2 = 0$.

3. *Least squares.* Note that for all $K \in \mathcal{C}$

$$\|A - K\|_{\mathbb{P}}^2 = \|A - \mathbb{P}(A|\mathcal{C}) + \mathbb{P}(A|\mathcal{C}) - K\|_{\mathbb{P}}^2 = \|A - \mathbb{P}(A|\mathcal{C})\|_{\mathbb{P}}^2 + \|\mathbb{P}(A|\mathcal{C}) - K\|_{\mathbb{P}}^2 \geq \|A - \mathbb{P}(A|\mathcal{C})\|_{\mathbb{P}}^2,$$

where, in the next to last step, we used that $A - \mathbb{P}(A|\mathcal{C})$ is orthogonal to $\mathbb{P}(A|\mathcal{C}) - K \in \mathcal{C}$.

□

Remark. The usual elementary properties of classical conditional expectations and their proofs [54] carry over directly to the noncommutative situation. In particular, we have linearity, positivity, invariance of the state $\mathbb{P}(\mathbb{P}(A|\mathcal{C})) = \mathbb{P}(A)$, invariance of \mathcal{C} ($\mathbb{P}(A|\mathcal{C}) = A$ if $A \in \mathcal{C}$), the tower property $\mathbb{P}(\mathbb{P}(A|\mathcal{B})|\mathcal{C}) = \mathbb{P}(A|\mathcal{C})$ if $\mathcal{C} \subset \mathcal{B}$, the module property $\mathbb{P}(AB|\mathcal{C}) = B\mathbb{P}(A|\mathcal{C})$ for $B \in \mathcal{C}$, etc. As an example, let us prove linearity. It suffices to show that $Z = \alpha\mathbb{P}(A|\mathcal{C}) + \beta\mathbb{P}(B|\mathcal{C})$ satisfies the

definition of $\mathbb{P}(\alpha A + \beta B | \mathcal{C})$, i.e. $\mathbb{P}(ZC) = \mathbb{P}((\alpha A + \beta B)C)$ for all $C \in \mathcal{C}$. But this is immediate from the linearity of \mathbb{P} and Definition 2.2.

We reemphasize that the conditional expectation $\mathbb{P}[A | \mathcal{C}]$ is a *least squares estimate* of A . Hence the quantum filtering problem is essentially an estimation problem, just like its classical counterpart. We can extend the conditional expectation, as we did for the state \mathbb{P} , to be defined for any self-adjoint operator that is affiliated to \mathcal{A} . This way the least squares estimate given a set of observations is defined for any quantum observable that is compatible with these observations.

3 Stochastic processes on Fock space

After having briefly discussed the framework of noncommutative probability theory, we now turn to one particular quantum probability space. Within this model we will discover many interesting classical stochastic processes, i.e. a whole family of Wiener processes and Poisson processes. However, these processes do not commute amongst each other. An extension of Itô's stochastic calculus, due to Hudson and Parthasarathy [35], unites all these processes again in one noncommutative stochastic calculus (see Section 4). In Section 5 we shall argue that the model studied here appropriately describes the quantised electromagnetic field and its interaction with matter.

Let \mathcal{H} be a Hilbert space. The *symmetric* or *Bosonic Fock space* over \mathcal{H} is defined as

$$\mathcal{F}(\mathcal{H}) := \mathbb{C} \oplus \bigoplus_{n=1}^{\infty} \mathcal{H}^{\otimes n}.$$

We will take \mathcal{H} to be $L^2(\mathbb{R})$, the space of quadratically integrable functions on \mathbb{R} , and denote $\mathcal{F}(L^2(\mathbb{R}))$ simply by \mathcal{F} . The Fock space \mathcal{F} is closely related to the Wiener chaos expansion in probability theory [46, 15] and from a physics point of view it describes a field of bosonic particles, like photons. Then the term $L^2(\mathbb{R})^{\otimes n}$ in the direct sum defining \mathcal{F} , describes the situation where there are n photons present. Since photons are bosons they have to be described by symmetric wavefunctions, explaining the symmetric tensor product in the definition.

For every $f \in \mathcal{H}$ we define the *exponential vector* $e(f) \in \mathcal{F}$ by

$$e(f) := 1 \oplus \bigoplus_{n=1}^{\infty} \frac{1}{\sqrt{n!}} f^{\otimes n}. \quad (1)$$

The linear span \mathcal{D} of all exponential vectors is a dense subspace of \mathcal{F} . On the dense domain \mathcal{D} we define for all $f \in \mathcal{H}$ an operator $W(f)$ by

$$W(f)e(g) := \exp\left(-\langle f, g \rangle - \frac{1}{2}\|f\|^2\right)e(f+g), \quad g \in \mathcal{H}. \quad (2)$$

These operators are isometric and therefore uniquely extend to unitary operators, also denoted $W(f)$, on \mathcal{F} . The operators $W(f) : \mathcal{F} \rightarrow \mathcal{F}$ are called *Weyl operators* and they satisfy the following *Weyl relations*

$$\begin{aligned} 1. \quad & W(f)^* = W(-f), \quad f \in \mathcal{H}, \\ 2. \quad & W(f)W(g) = \exp(-i\text{Im}\langle f, g \rangle)W(f+g), \quad f, g \in \mathcal{H}. \end{aligned} \quad (3)$$

It can be shown [47] that the Weyl operators $W(f)$ ($f \in \mathcal{H}$) generate the von Neumann algebra of all bounded operators on \mathcal{F} , i.e. $\text{vN}(W(f); f \in \mathcal{H}) = \mathcal{B}(\mathcal{F}) =: \mathcal{W}$. It follows from the Weyl relations that for all $f \in \mathcal{H}$ the family $\{W(tf)\}_{t \in \mathbb{R}}$ forms a one-parameter group of unitary operators on \mathcal{F} . It can be shown to be continuous in the strong operator topology [47]. The following theorem is a classic result in spectral theory and can be found for instance in [38].

Theorem 3.1: (Stone's theorem) *Let $\{U_t\}_{t \in \mathbb{R}}$ be a group of unitary operators in some von Neumann algebra \mathcal{A} , continuous in the strong operator topology. There exists a unique self-adjoint operator A affiliated to \mathcal{A} such that*

$$U_t = \exp(itA) := \int_{\mathbb{R}} e^{it\lambda} E(d\lambda),$$

where E denotes the spectral measure of the self-adjoint operator A .

Since $\{W(tf)\}_{t \in \mathbb{R}}$ is a strongly continuous one-parameter group of unitaries, Stone's theorem provides a self-adjoint operator $B(f)$ such that

$$W(tf) = \exp(itB(f)).$$

The operators $B(f)$ ($f \in \mathcal{H}$) are called *field operators*. The domain of the operator $B(f_k) \dots B(f_1)$ contains \mathcal{D} for every $f_1, \dots, f_k \in \mathcal{H}$ and $k \in \mathbb{N}$ (cf. [48]). For $f, g \in \mathcal{H}$ and $t \in \mathbb{R}$ it follows from the Weyl relations that on the domain \mathcal{D}

$$\begin{aligned} 1. \quad & B(tf) = tB(f), \\ 2. \quad & B(f+g) = B(f) + B(g), \\ 3. \quad & [B(f), B(g)] = 2i\text{Im}\langle f, g \rangle. \end{aligned} \tag{4}$$

The last relation (4.3) is called the *canonical commutation relation*. We have introduced it via the Weyl operators since they have the advantage of being bounded.

We fix an α in $[0, \pi)$ and denote by H_α the subspace of $L^2(\mathbb{R})$ of functions of the form $e^{i\alpha}f$ with f a real valued function in $L^2(\mathbb{R})$. For $f \in H_\alpha$ we define, as before, bounded operators $T(f)_+ := (B(f) + iI)^{-1}$ and $T(f)_- := (B(f) - iI)^{-1}$. It follows from the canonical commutation relation that the family $\{T(f)_+, T(f)_-; f \in H_\alpha\}$ is commutative and therefore generates a commutative von Neumann algebra \mathcal{C}_α . We denote by ϕ the *vacuum state* on $\mathcal{W} = \mathcal{B}(\mathcal{F})$, given by $\phi(X) := \langle \Phi, X\Phi \rangle$ where $\Phi := e(0) = 1 \oplus 0 \oplus 0 \oplus \dots \in \mathcal{F}$ is the *vacuum vector*. Using Theorem 2.1 we obtain a probability space $(\Omega_\alpha, \Sigma_\alpha, \mathbf{P}_\alpha)$ and a *-isomorphism $\iota: \mathcal{C}_\alpha \rightarrow L^\infty(\Omega_\alpha, \Sigma_\alpha, \mathbf{P}_\alpha)$ such that $\phi(C) = \mathbb{E}_{\mathbf{P}_\alpha}[\iota(C)]$ for all $C \in \mathcal{C}_\alpha$. In a similar fashion as before we can now define $\iota(B(f))$ for the (possibly unbounded) self-adjoint operator $B(f)$.

Let us study the random variable $\iota(B(f))$ on the probability space $(\Omega_\alpha, \Sigma_\alpha, \mathbf{P}_\alpha)$ in some more detail by examining its characteristic function

$$\mathbb{E}_{\mathbf{P}_\alpha} \left[\exp \left(ix \iota(B(f)) \right) \right] = \phi \left(\exp \left(iB(xf) \right) \right) = \left\langle \Phi, W(xf)\Phi \right\rangle = \exp \left(- \frac{x^2 \|f\|^2}{2} \right), \quad x \in \mathbb{R}. \tag{5}$$

In the last step we used the definition of the Weyl operator, equation (2), and the relation $\langle e(f), e(g) \rangle = \exp \langle f, g \rangle$, which easily follows from equation (1). Define a random process on

$(\Omega_\alpha, \Sigma_\alpha, \mathbf{P}_\alpha)$ by

$$W_t^\alpha := \iota\left(B(e^{i\alpha}\chi_{[0,t]})\right), \quad t \geq 0,$$

where $\chi_{[0,t]}$ denotes the indicator function of the interval $[0, t]$. From the characteristic functional $f \mapsto \mathbb{E}_{\mathbf{P}_\alpha} \left[\exp \left(i\iota(B(f)) \right) \right]$ ($f \in H_\alpha$) given by equation (5), it follows that W_t^α is a process with independent, normally distributed increments $W_t^\alpha - W_s^\alpha$ ($s \leq t$) such that their means are zero and their variances are $t - s$. Summarizing, W_t^α is a *Wiener process* on $(\Omega_\alpha, \Sigma_\alpha, \mathbf{P}_\alpha)$.

Identifying the operators $B(f)$ and the random variables $\iota(B(f))$, we see that the algebra \mathcal{W} of all bounded operators on the Fock space \mathcal{F} contains a whole family (indexed by $\alpha \in [0, \pi)$) of Wiener processes. Note however, that these processes for different values of α , do not commute. For example, it follows from the canonical commutation relation (4.3) that $[W_s^0, W_t^{\pi/2}] = 2i \min\{s, t\} \neq 0$. Therefore, for different values of α , these processes can not be represented simultaneously on the same probability space via the spectral theorem. It is precisely in this sense that noncommutative probability is richer than classical probability.

The idea to simultaneously diagonalise the fields in the family $\{B(e^{i\alpha}\chi_{[0,t]}); t \geq 0\}$ is implicit in some of the earliest work in quantum field theory. However, Segal [49] in the 1950s was the first to emphasise the connection with probability theory. Apart from the Wiener processes W_t^α , the algebra \mathcal{W} contains Poisson processes. Before introducing them, some further preparations have to be made.

The *second quantisation* of an operator $A \in \mathcal{B}(\mathcal{H})$ is the operator $\Gamma(A) \in \mathcal{W} = \mathcal{B}(\mathcal{F})$ defined by

$$\Gamma(A) := I \oplus \bigoplus_{n=1}^{\infty} A^{\otimes n}.$$

For all $A, B \in \mathcal{B}(\mathcal{H})$ this immediately gives $\Gamma(AB) = \Gamma(A)\Gamma(B)$. Let S be a self-adjoint element in $\mathcal{B}(\mathcal{H})$, then $\exp(itS)$ is a one-parameter group of unitaries in $\mathcal{B}(\mathcal{H})$. After second quantisation, this leads to a one-parameter group $\Gamma(\exp(itS))$ of unitaries in \mathcal{W} (continuous in the strong operator topology). Stone's theorem 3.1 then asserts the existence of a self-adjoint operator $\Lambda(S)$ on \mathcal{F} such that for all $t \in \mathbb{R}$

$$\Gamma(\exp(itS)) = \exp(it\Lambda(S)).$$

The domain of a product $\Lambda(S_1) \dots \Lambda(S_n)$ contains \mathcal{D} for all self-adjoint elements S_1, \dots, S_n in $\mathcal{B}(\mathcal{H})$ [47]. Denote by P_t the projection $L^2(\mathbb{R}) \rightarrow L^2(\mathbb{R}) : f \mapsto \chi_{[0,t]}f$ where $\chi_{[0,t]}$ denotes the indicator function of the interval $[0, t]$. For notational convenience, we abbreviate the operator $\Lambda(P_t)$ to $\Lambda(t)$. On the n th-layer of the symmetric Fock space $\Gamma(\exp(isP_t))$ acts as $\exp(isP_t)^{\otimes n}$. Differentiation with respect to s shows that on the n th-layer of the symmetric Fock space $\Lambda(t) = P_t \otimes I^{\otimes n-1} + I \otimes P_t \otimes I^{\otimes n-2} + \dots + I^{\otimes n-1} \otimes P_t$. This shows that $\Lambda(t)$ is the operator that counts how many particles, i.e. photons in the field, are present in the interval $[0, t]$. In particular we therefore have $\Lambda(t)\Phi = 0$, a property that we will exploit later on.

Since the family of projections $\{P_t; t \geq 0\}$ is commutative, it generates a commutative von Neumann algebra \mathcal{N} . For all self-adjoint elements S and T in \mathcal{N} we have on the dense domain \mathcal{D}

$$[\Lambda(S), \Lambda(T)] = \lim_{t \rightarrow 0} \frac{e^{it\Lambda(T)} e^{it\Lambda(S)} e^{-it\Lambda(T)} e^{-it\Lambda(S)} - I}{t^2} = \lim_{t \rightarrow 0} \frac{\Gamma(e^{itT} e^{itS} e^{-itT} e^{-itS}) - I}{t^2} = 0.$$

For all self-adjoint $S \in \mathcal{N}$ we can define commuting bounded operators $T(S)_+ := (\Lambda(S) + iI)^{-1}$ and $T(S)_- := (\Lambda(S) - iI)^{-1}$. The above ensures that they generate a commutative von Neumann subalgebra \mathcal{C} of \mathcal{W} . For $f \in L^2(\mathbb{R})$ we define a *coherent vector* $\psi(f)$ by

$$\psi(f) := W(f)\Phi = W(f)e(0) = \exp\left(-\frac{\|f\|^2}{2}\right)e(f).$$

A *coherent state* ρ on \mathcal{W} is defined by $\rho(A) := \langle \psi(f), A\psi(f) \rangle$. The spectral theorem 2.1 provides a classical probability space $(\Omega, \Sigma, \mathbf{P})$ and a *-isomorphism $\iota : \mathcal{C} \rightarrow L^\infty(\Omega, \Sigma, \mathbf{P})$ such that $\rho(C) = \mathbb{E}_{\mathbf{P}}[\iota(C)]$. For all self-adjoint operators S in the von Neumann algebra \mathcal{N} , we can define as before random variables $\iota(\Lambda(S))$ on $(\Omega, \Sigma, \mathbf{P})$.

Let us investigate the characteristic function of the random variable $\iota(\Lambda(S))$ in some more detail

$$\begin{aligned} \mathbb{E}_{\mathbf{P}}\left[e^{ix\iota(\Lambda(S))}\right] &= \left\langle \psi(f), e^{ix\Lambda(S)}\psi(f) \right\rangle = \left\langle \psi(f), \Gamma(e^{ixS})\psi(f) \right\rangle \\ &= e^{-\|f\|^2} \left\langle e(f), e(e^{ixS}f) \right\rangle = e^{\langle f, (e^{ixS} - I)f \rangle}. \end{aligned}$$

The functional $S \mapsto \mathbb{E}_{\mathbf{P}}\left[e^{i\iota(\Lambda(S))}\right]$ shows that the process $N_t := \iota(\Lambda(t))$ is a Poisson process [35] on $(\Omega, \Sigma, \mathbf{P})$ with intensity measure $|f|^2 d\lambda$, where λ stands for the Lebesgue measure. Summarizing, when photons are counted in a laser beam they arrive Poisson distributed. Since $\psi(f) = W(f)e(0) = W(f)\Phi$, we could just as well have studied the commutative von Neumann algebra $W(f)^*\mathcal{C}W(f)$, equipped with the vacuum state ϕ . In this way we get a whole family of Poisson processes indexed by f within the quantum probability space (\mathcal{W}, ϕ) . Again, these processes do not commute amongst each other [35].

In this section we have seen that the quantum probability space (\mathcal{W}, ϕ) contains many interesting classical stochastic processes. However, these classical processes do not commute amongst each other. We will proceed by discussing the stochastic calculus of Hudson and Parthasarathy [35] enabling us to treat the stochastic analysis of all these processes in one framework.

4 Quantum stochastic calculus

We start this section with some technical definitions and manipulations to clear the way for defining the stochastic integrals of Hudson and Parthasarathy. However, it is not so much the definition of the stochastic integrals that is of the greatest importance here. It is the subsequent Itô rule obeyed by these stochastic integrals, summarized in their Itô table, that will enable us to put them to good use. The Itô rule translates the difficult analysis involved in defining the stochastic integrals into simple algebraic manipulations with increments. That is the real strength of having a stochastic calculus. We refer to [35, 15, 46, 47] for more extensive treatments of quantum stochastic calculus.

Let \mathcal{H}_1 and \mathcal{H}_2 be Hilbert spaces. The symmetric Fock space has the following *exponential property* (cf. [47])

$$\mathcal{F}(\mathcal{H}_1 \oplus \mathcal{H}_2) \cong \mathcal{F}(\mathcal{H}_1) \otimes \mathcal{F}(\mathcal{H}_2),$$

in the sense that there exists a unitary operator $U : \mathcal{F}(\mathcal{H}_1 \oplus \mathcal{H}_2) \rightarrow \mathcal{F}(\mathcal{H}_1) \otimes \mathcal{F}(\mathcal{H}_2)$ such that for all $f_1 \in \mathcal{H}_1$ and $f_2 \in \mathcal{H}_2$ the exponential vector $e(f_1 \oplus f_2)$ is mapped to the tensor

product $e(f_1) \otimes e(f_2)$. For $s \leq t$ and $f \in L^2(\mathbb{R})$ we write $f_{[t]} := \chi_{(-\infty, t]} f$, $f_{[t]} := \chi_{[t, \infty)} f$ and $f_{[s, t]} := \chi_{[s, t]} f$. Furthermore we have $L^2(\mathbb{R}) = L^2((-\infty, t]) \oplus L^2([t, \infty))$, which means that every f in $L^2(\mathbb{R})$ can be uniquely written as a sum $f = f_{[t]} + f_{[t]}$ of elements in $L^2((-\infty, t])$ and $L^2([t, \infty))$. Writing $\mathcal{F}_{[t]} := \mathcal{F}(L^2((-\infty, t]))$, $\mathcal{F}_{[t]} := \mathcal{F}(L^2([t, \infty)))$ and $\mathcal{F}_{[s, t]} := \mathcal{F}(L^2([s, t]))$, this leads to the splitting $\mathcal{F} = \mathcal{F}_{[t]} \otimes \mathcal{F}_{[t]}$ where, as before, exponential vectors of direct sums are identified with tensor products of exponential vectors. Since t can vary continuously through \mathbb{R} , \mathcal{F} is said to be a *continuous tensor product*. The algebra of all bounded operators on \mathcal{F} splits in a similar way, that is $\mathcal{W} = \mathcal{B}(\mathcal{F}) = \mathcal{B}(\mathcal{F}_{[t]}) \otimes \mathcal{B}(\mathcal{F}_{[t]}) = \mathcal{W}_{[t]} \otimes \mathcal{W}_{[t]}$ (cf. [47]), where we denote $\mathcal{W}_{[t]} := \mathcal{B}(\mathcal{F}_{[t]})$, $\mathcal{W}_{[t]} := \mathcal{B}(\mathcal{F}_{[t]})$ and $\mathcal{W}_{[s, t]} := \mathcal{B}(\mathcal{F}_{[s, t]})$.

On the domain \mathcal{D} we introduce *annihilation operators* A_t and *creation operators* A_t^* by

$$A_t := \frac{1}{2}(B(i\chi_{[0, t]}) - iB(\chi_{[0, t]})) \quad \text{and} \quad A_t^* := \frac{1}{2}(B(i\chi_{[0, t]}) + iB(\chi_{[0, t]})).$$

It can be shown [47] that $A_t e(f) = \langle \chi_{[0, t]}, f \rangle e(f)$ and that A_t^* is its adjoint on \mathcal{D} . This means that these operators are the annihilation and creation operators for the mode $\chi_{[0, t]}$ of the field as they are known in physics. Note that $A_t \Phi = A_t e(0) = 0$, a property that we will exploit in future. We will denote by Λ_t the restriction of $\Lambda(t)$ to the domain \mathcal{D} . It can be shown [47] that $\langle e(g), \Lambda_t e(f) \rangle = \langle g, \chi_{[0, t]} f \rangle \langle e(g), e(f) \rangle$. This implies the important relation $\Lambda_t \Phi = 0$ that we already encountered in the previous section.

Let M_t be one of the processes A_t , A_t^* or Λ_t . The following *factorisation property* [35, 47] underlies the definition of the stochastic integral

$$(M_t - M_s)e(f) = e(f_{[s, t]}) \left((M_t - M_s)e(f_{[s, t]}) \right) e(f_{[t]}), \quad s \leq t,$$

such that $(M_t - M_s)e(f_{[s, t]}) \in \mathcal{F}_{[s, t]}$. We have made our notation lighter by omitting the tensor product signs. Let \mathcal{H} be a Hilbert space, called the *initial space*. We tensor the initial space to \mathcal{F} and extend the operators A_t , A_t^* and Λ_t to $\mathcal{H} \otimes \mathcal{F}$ by ampliation, i.e. by tensoring the identity to them on \mathcal{H} (however, to keep notation light we will not denote it). Just for mathematical convenience we will take \mathcal{H} to be finite dimensional, i.e. $\mathcal{H} = \mathbb{C}^n$. We denote the algebra of all operators on \mathcal{H} by \mathcal{B} . We are ready for the definition of the stochastic integral.

Definition 4.1: (Quantum stochastic integral) Let $\{L_s\}_{0 \leq s \leq t}$ be an adapted (i.e. $L_s \in \mathcal{B} \otimes \mathcal{W}_s$) for all $0 \leq s \leq t$) simple process with respect to the partition $\{s_0 = 0, s_1, \dots, s_p = t\}$ in the sense that $L_s = L_{s_j}$ whenever $s_j \leq s < s_{j+1}$. The stochastic integral of L with respect to M on $\mathbb{C}^n \otimes \mathcal{D}$ is then defined as [35, 47]

$$\int_0^t L_s dM_s x e(f) := \sum_{j=0}^{p-1} \left(L_{s_j} x e(f_{[s_j]}) \right) \left((M_{s_{j+1}} - M_{s_j}) e(f_{[s_j, s_{j+1}]}) \right) e(f_{[s_{j+1}]}) , \quad x \in \mathbb{C}^n.$$

The notation is simplified by writing $dX_t = L_t dM_t$ for $X_t = X_0 + \int_0^t L_s dM_s$. The definition of the stochastic integral can be extended to a large class of stochastically integrable processes [35, 47] if we approximate these by simple functions and take a limit in the strong operator topology.

Since $A_t \Phi = \Lambda_t \Phi = 0$ it is immediate from the definition that quantum stochastic integrals with respect to A_t and Λ_t acting on Φ are zero, or infinitesimally $d\Lambda_t \Phi_t = dA_t \Phi_t = 0$. From this we

can immediately conclude that vacuum expectations of stochastic integrals with respect to A_t and Λ_t vanish. Furthermore, we have $\langle \Phi, \int_0^t L_s dA_s^* \Phi \rangle = \langle \int_0^t L_s^* dA_s \Phi, \Phi \rangle = 0$, i.e. vacuum expectation of stochastic integrals with respect to A_t^* are zero as well. Note, however, that $dA_t^* \Phi|_t \neq 0$.

To get some more feeling for the definition of the quantum stochastic integral we will now investigate which quantum stochastic differential equation is satisfied by the Weyl operators $W(f_t)$ ($f \in L^2(\mathbb{R})$). Note that the stochastic integrals are defined on the domain \mathcal{D} . Therefore we calculate for g and h in $L^2(\mathbb{R})$

$$\phi(t) := \langle e(g), W(f_t)e(h) \rangle = e^{-\langle f_t, h \rangle - \frac{1}{2}\|f_t\|^2} \langle e(g), e(h + f_t) \rangle = e^{\langle g, f_t \rangle - \langle f_t, h \rangle - \frac{1}{2}\|f_t\|^2} e^{\langle g, h \rangle},$$

which means that

$$\phi(t) - \phi(0) = \int_0^t \left\langle e(g), \frac{d}{ds} \left(\langle g, f_s \rangle - \langle f_s, h \rangle - \frac{1}{2}\|f_s\|^2 \right) \phi(s) e(h) \right\rangle ds.$$

Let us turn to the definition of the stochastic integral, Definition 4.1. Let $\{0 = s_0, s_1, \dots, s_p = t\}$ be a partition of $[0, t]$ and choose $L_s = \bar{f}(s_j)W(f_{s_j})$ for $s_j \leq s < s_{j+1}$. Let further M_t be A_t , then the definition of the stochastic integral gives (heuristically in the last step)

$$\begin{aligned} & \sum_{j=0}^{p-1} \left(\bar{f}(s_j)W(f_{s_j})e(h_{s_j}) \right) \left((A_{s_{j+1}} - A_{s_j})e(h_{[s_j, s_{j+1}]}) \right) e(h_{[s_{j+1}]}) = \\ & \sum_{j=0}^{p-1} \left\langle f(s_j)\chi_{[s_j, s_{j+1}]}, h \right\rangle W(f_{s_j})e(h) = \sum_{j=0}^{p-1} \left(\left\langle f(s_j)\chi_{[s_j, s_{j+1}]}, h \right\rangle - \left\langle f(s_j)\chi_{s_j}, h \right\rangle \right) W(f_{s_j})e(h) \\ & \longrightarrow \int_0^t d\langle f_s, h \rangle W(f_s) e(h). \end{aligned}$$

Together with a similar calculation for $M_t = A_t^*$, this yields the following quantum stochastic differential equation for the Weyl operator $W(f_t)$

$$dW(f_t) = \left\{ f(t)dA_t^* - \bar{f}(t)dA_t - \frac{1}{2}|f(t)|^2 dt \right\} W(f_t). \quad (6)$$

Let us return to developing the theory further. To be able to compute with the stochastic integral, we need a stochastic calculus, i.e. a quantum Itô rule. The basic ingredient for its proof comes from the commutation relations between A_t , A_t^* and Λ_t . For instance, from the canonical commutation relation and the definition of A_t and A_t^* we see $[A_t, A_t^*] = t$ on the exponential domain \mathcal{D} . Therefore we have

$$\langle e(f), A_t A_t^* e(g) \rangle = \langle A_t e(f), A_t e(g) \rangle + t \langle e(f), e(g) \rangle = \left(\langle f, \chi_{[0, t]} \rangle \langle \chi_{[0, t]}, g \rangle + t \right) \langle e(f), e(g) \rangle.$$

Infinitesimally this immediately leads to $d(A_t^* A_t) = A_t^* dA_t + A_t dA_t^* + dt$. The next theorem is built on similar ideas, it can be found in [35] and [47].

Theorem 4.2: (Quantum Itô rule [35]) Let X_t and Y_t be stochastic integrals of the form

$$\begin{aligned} dX_t &= B_t d\Lambda_t + C_t dA_t + D_t dA_t^* + E_t dt, \\ dY_t &= F_t d\Lambda_t + G_t dA_t + H_t dA_t^* + I_t dt, \end{aligned}$$

for some stochastically integrable processes $B_t, C_t, D_t, E_t, F_t, G_t, H_t$ and I_t (see [35, 47]). The process $X_t Y_t$ then satisfies the relation

$$d(X_t Y_t) = X_t dY_t + (dX_t) Y_t + dX_t dY_t,$$

on the domain \mathcal{D} , where $dX_t dY_t$ should be evaluated according to the quantum Itô table

	dA_t	$d\Lambda_t$	dA_t^*	dt
dA_t	0	dA_t	dt	0
$d\Lambda_t$	0	$d\Lambda_t$	dA_t^*	0
dA_t^*	0	0	0	0
dt	0	0	0	0

$$\text{i.e. } dX_t dY_t = B_t F_t d\Lambda_t + C_t F_t dA_t + B_t H_t dA_t^* + C_t H_t dt.$$

In the previous section we encountered the classical Wiener processes $B(e^{i\alpha}\chi_{[0,t]}) = ie^{-i\alpha}A_t - ie^{i\alpha}A_t^*$ for $\alpha \in [0, \pi)$. Since $dB(e^{i\alpha}\chi_{[0,t]}) = ie^{-i\alpha}dA_t - ie^{i\alpha}dA_t^*$, we recover the classical Itô rule for these Wiener processes, i.e. $(dB(e^{i\alpha}\chi_{[0,t]}))^2 = dt$, from the quantum Itô rule, as it should. For an $f \in L^2(\mathbb{R})$ we can write the Weyl operator $W(f_t)$ as $W(f_t) = \exp(\int_0^t f(s)d(A_s^* - A_s))$. Therefore it follows from the quantum Itô rule that the Weyl operators $W(f_t)$ satisfy equation (6), where $-\frac{1}{2}\|f\|^2 W(f_t)dt$ is the Itô correction term. The Poisson process of the previous section was given by $\Lambda_t^f := W(f)^* \Lambda_t W(f)$ in the vacuum state, for which the quantum Itô rule gives

$$d\Lambda_t^f = d\Lambda_t + \overline{f}(t)dA_t + f(t)dA_t^* + |f(t)|^2 dt,$$

which leads to the classical Itô rule $(d\Lambda_t^f)^2 = d\Lambda_t^f$ for the Poisson process. Furthermore, integrating the above equation, we see that $\Lambda_t^f = \Lambda_t + B(if_t) + \int_0^t |f(s)|^2 ds$.

In the next section, we will show that quantum stochastic differential equations (QSDEs) of the following form

$$dU_t = \left\{ LdA_t^* - L^*dA_t - \frac{1}{2}L^*Ldt - iHdt \right\} U_t, \quad U_0 = I, \quad (7)$$

where L and H are in \mathcal{B} and H is self-adjoint, emerge naturally from physical models. A standard Picard iteration argument [35, 47] ensures existence and uniqueness of the solution. The adjoint U_t^* satisfies

$$dU_t^* = U_t^* \left\{ L^*dA_t - LdA_t^* - \frac{1}{2}L^*Ldt + iHdt \right\}, \quad U_0^* = I.$$

From the quantum Itô rule it now immediately follows that $d(U_t^* U_t) = 0$, which means that the solution U_t is unitary for all t . The interpretation is that U_t defines a time evolution or *flow* $X \mapsto U_t^* X U_t$ in $\mathcal{B} \otimes \mathcal{W}$, i.e., an observation of X at time t is described by the observable $U_t^* X U_t$.

Remark. Despite the fact that both $A_t + A_t^*$ and $i(A_t - A_t^*)$ are classical noises, Eq. (7) is not a classical stochastic differential equation. To make this idea explicit, rewrite the equation as

$$dU_t = \left\{ iL_+ i(dA_t - dA_t^*) + iL_-(dA_t + dA_t^*) - \frac{1}{2}L^*Ldt - iHdt \right\} U_t, \quad U_0 = I, \quad (8)$$

where $L_+ = (L + L^*)/2$ and $L_- = (L - L^*)/(2i)$. This reveals particularly well that the initial system is driven simultaneously by two noncommuting noises. If either $L_+ = 0$ or $L_- = 0$, we say that U_t is *essentially commutative* [41].

Define $T_t(X) := \text{id} \otimes \phi(U_t^* X \otimes I U_t)$ for all $X \in \mathcal{B}$ and $t \geq 0$. Using the quantum Itô rule and the fact that vacuum expectations of stochastic integrals vanish, we find for all $X \in \mathcal{B}$

$$dT_t(X) = \text{id} \otimes \phi(d(U_t^* X \otimes I U_t)) = T_t(\mathcal{L}_{L,H}(X))dt,$$

where we have defined the Lindblad generator [44]

$$\mathcal{L}_{L,H}(X) = i[H, X] + L^* X L - \frac{1}{2}(L^* L X + X L^* L), \quad X \in \mathcal{B}.$$

In quantum probability, this object plays the same role as the infinitesimal generator of a Markov diffusion in classical probability theory. Indeed, the semigroup T_t , describing the time evolution of the expectation of any system observable, can be written as $T_t = \exp(t\mathcal{L}_{L,H})$. The commutator with H describes the unitary evolution of the initial system itself, whereas the residual terms describe the irreversible effect that the interaction with the environment has on the initial system.

A more general case can be treated in a similar fashion by introducing more channels in the field. That is, the initial system \mathbb{C}^n is tensored to k Fock spaces. The QSDE that defines the time evolution of the initial system and the field together is given in a general form by

$$dU_t = \{L_j dA_j^*(t) + (S_{ij} - \delta_{ij})d\Lambda_{ij}(t) - L_i^* S_{ij} dA_j(t) - (iH + \frac{1}{2}L_j^* L_j)dt\}U_t, \quad U_0 = I,$$

where repeated indices are being summed. The index j on $A_j(t)$ and $A_j(t)^*$ labels the annihilator and creator on the j th copy of the Fock space, S is a unitary operator on $\mathbb{C}^n \otimes l^2(\{1, 2, \dots, k\})$ such that $S_{ij} = \langle i, S j \rangle$, and $\Lambda_{ij}(t) := \Lambda(P_t \otimes |i\rangle\langle j|)$. Physically, the term $(S_{ij} - \delta_{ij})d\Lambda_{ij}(t)$ describes direct scattering between channels i and j in the field [7]. For this equation we again have existence, uniqueness and unitarity of its solution [47]. Moreover, we obtain the associated generator

$$\mathcal{L}(X) = i[H, X] + \sum_{j=1}^k L_j^* X L_j - \frac{1}{2}\{L_j^* L_j, X\}, \quad X \in \mathcal{B}, \quad (9)$$

where $\{X, Y\}$ stands for the anti-commutator $XY + YX$. It was shown by Lindblad [44] that any semigroup of completely positive identity preserving operators on $\mathcal{B} = M_n(\mathbb{C})$ has a generator of the form Eq. (9).

5 The filtering problem in quantum optics

Beside the intrinsic interest of quantum probability as a mathematical generalization of classical probability theory, many realistic physical scenarios are very well described by QSDEs of the form we have discussed. The inherent stochasticity of quantum mechanical problems makes this field a rich playground for the application of the classical theories of statistical inference and control of stochastic processes. Of course, as in the classical theory, white noise systems are only an idealization of physical interactions; a Markov limit of wide-band noise in the spirit of Wong and

Zakai gives stochastic models in the Itô form. For a large class of quantum systems, particularly those arising in the field of quantum optics, such approximations are extremely good and describe laboratory experiments essentially to experimental precision. Though a detailed discussion of the physics involved in the modelling of such systems is beyond the scope of these notes, we here briefly describe the physical origin of the equations that are widely used in the physics community [26].

The basic model of quantum optics consists of some fixed physical system, e.g. a collection of atoms, in interaction with the electromagnetic field. The atomic observables are self-adjoint operators in a Hilbert space \mathcal{H} . The description of the electromagnetic field and its interaction with the atoms follows from basic physical arguments (e.g. quantization of Maxwell's equations; see the excellent monograph [20] for a thorough treatment of this field, known as *quantum electrodynamics*.) It turns out that the free electromagnetic field, i.e. an optical field in empty space, is described by a stationary Gaussian (noncommutative) wide band noise $\tilde{a}(t, \mathbf{r})$ that propagates through space at the speed of light c ; i.e. if we restrict ourselves to a single spatial dimension, $\tilde{a}(t+\tau, z) = \tilde{a}(t, z-c\tau)$. If we now place the atoms at the origin $z = 0$, then the quantum dynamics is given by a Schrödinger equation of the form

$$\frac{d}{dt}\tilde{U}(t) = -\frac{i}{\hbar} [H - i\hbar(L(t) + L^*(t))(\tilde{a}(t, 0) - \tilde{a}^*(t, 0))] \tilde{U}(t),$$

where $L(t)$ is an atomic (dipole) operator and H is an atomic Hamiltonian. This equation, which follows directly from the physical model, is similar to (7) but has wide-band right hand side. One generally assumes that the dipole operator is harmonic, $L(t) = Le^{-i\omega t}$, but this assumption can be relaxed [3]. Hence L and H are operators in the initial system \mathcal{B} , H being self-adjoint.

We now wish to approximate this equation by an equation where the noise has infinite bandwidth. There is a large body of literature on such approximations in classical probability, following the pioneering work of Wong and Zakai (see e.g. [42] and references therein.) A common way to attain this limit is to rescale the (wide-band) noise term by $1/\varepsilon$, time as t/ε^2 , and then take the limit $\varepsilon \rightarrow 0$. The effect of this rescaling [43] is that the noise bandwidth goes to infinity, whereas its energy per unit bandwidth is retained. A similar intuition holds in the noncommutative case; i.e., one can show that for the type of model we have described

$$\frac{1}{\varepsilon} \int_0^t L(s/\varepsilon^2)(\tilde{a}^*(s/\varepsilon^2, 0) - \tilde{a}(s/\varepsilon^2, 0)) ds \longrightarrow \sqrt{\gamma} LA_t^* \quad \text{as } \varepsilon \longrightarrow 0 \quad (\text{in distribution}),$$

where $\gamma > 0$ depends on the characteristics of the wide-band noise \tilde{a} and on the dipole rotation frequency ω [2, 3, 32].

The essential question is now the limiting behavior as $\varepsilon \rightarrow 0$ of the rescaled Schrödinger equation

$$\frac{d}{dt}\tilde{U}^\varepsilon(t) = -\frac{i}{\hbar} [H - i\hbar(L(t/\varepsilon^2) + L^*(t/\varepsilon^2))(\tilde{a}(t/\varepsilon^2, 0) - \tilde{a}^*(t/\varepsilon^2, 0)) / \varepsilon] \tilde{U}^\varepsilon(t). \quad (10)$$

The limit $\varepsilon \rightarrow 0$ of this equation is known as the *weak coupling limit*, or the Friedrichs-Van Hove limit, in the quantum probability literature. The limit can be studied by expanding the solution of Eq. (10) in a (Dyson) series using Picard iteration, and then calculating the limit of each term in the series [2, 3, 32, 31]. In particular, as we are interested in the time evolution $\tilde{U}(t)^* X(t) \tilde{U}(t)$ of any observable X , one can study the convergence of each term in the series expansion of $\tilde{U}(t)^{\varepsilon*} X \tilde{U}^\varepsilon(t)$. One finds that $\tilde{U}(t)^{\varepsilon*} X \tilde{U}^\varepsilon(t) \rightarrow U_t^* X U_t$ as $\varepsilon \rightarrow 0$, where

$$dU_t = \left\{ \sqrt{\gamma} L dA_t^* - \sqrt{\gamma} L^* dA_t - \frac{1}{2}(\gamma + i\sigma) L^* L dt - iH dt \right\} U_t, \quad U_0 = I,$$

with $\sigma \in \mathbb{R}$. The additional term $\propto L^*L$ is the Itô correction term obtained in the limit. It is a signature of the noncommutativity of the wide band noise that the constant multiplying the Itô correction is complex; the correction consists of a diffusive term $\gamma L^*L/2$ and a Hamiltonian term $i\sigma L^*L/2$. In the following we will absorb the constant γ into the definition of L , i.e. $\sqrt{\gamma}L \mapsto L$, and the Hamiltonian correction into the definition of H , i.e. $H + \sigma L^*L/2 \mapsto H$. This gives

$$dU_t = \left\{ LdA_t^* - L^*dA_t - \frac{1}{2}L^*Ldt - iHdt \right\} U_t, \quad U_0 = I, \quad (11)$$

which is a QSDE in the Hudson-Parthasarathy form like we encountered previously. Henceforth we will take this equation as our physical model. We note that in some situations it is also possible to obtain gauge ($d\Lambda_t$) processes in the weak coupling limit [32]; this is particularly useful for the description of a strong off-resonant laser probe, which is essentially a scattering interaction.

In classical models, the system dynamics is usually described by an SDE which determines the time evolution of the state of the system. A classical “system observable”, then, is any function $f(x_t)$ of the system state x_t . The flow j_t of the dynamics then returns the random variable $j_t(f) = f(x_t)$ corresponding to the observable f at time t . Due to the fact that quantum models are noncommutative we cannot give a sample path interpretation to the full model; in particular, there is no analogue of the system state x_t in quantum mechanics. On the other hand, we do have a well-defined concept of a system observable, which is just any observable of the initial system. The unitary solution U_t of Eq. (11) then provides a quantum analogue of the flow j_t , given by $j_t(X) = U_t^* X U_t$. This description of a stochastic model by a fixed probability measure (state) and time-varying random variables (observables), which is nearly universal in classical stochastics, is known as the *Heisenberg picture* in quantum mechanics. It is not difficult, using the Itô rules, to find a dynamical equation for j_t

$$dj_t(X) = j_t(\mathcal{L}_{L,H}(X))dt + j_t([L^*, X])dA_t + j_t([X, L])dA_t^*. \quad (12)$$

This is the quantum analogue of the classical Itô formula for $j_t(f)$.

Having described the dynamics of the system and its interaction with the field, let us now turn to the observations that we can perform. Unlike in classical stochastic theory, where one observes some observable of the system, in quantum models an observation is generally performed in the field. From the system’s perspective, the interaction with the field looks like an (albeit noncommutative) noisy driving force. Similarly, however, the field is perturbed by its interaction with the atoms, and carries off information as it propagates away after the interaction. By performing a measurement in the field, then, we can attempt to perform statistical inference of the atomic observables.

To calculate the perturbation of the field by the atoms we once again calculate $U_t^* Y U_t$, where now, however, Y is a field observable. The field observable of interest depends on the type of measurement we choose to perform. Without entering into the details, we mention two types of measurement that are extremely common in quantum optics: direct photodetection (photon counting), for which the observation at time t is given by $Y_t^\Lambda = U_t^* \Lambda_t U_t$, and homodyne detection, for which $Y_t^W = U_t^* (A_t + A_t^*) U_t$ (more generally $Y_t^W = U_t^* (e^{-i\varphi} A_t + e^{i\varphi} A_t^*) U_t$.) We refer to [6, 5] for a detailed treatment of quantum optical measurements. Using the Itô rules we obtain

$$\begin{aligned} dY_t^\Lambda &= d\Lambda_t + j_t(L) dA_t^* + j_t(L^*) dA_t + j_t(L^*L) dt, \\ dY_t^W &= j_t(L + L^*) dt + dA_t + dA_t^*. \end{aligned}$$

Intuitively, it appears that Y_t^Λ is like a Poisson process whose intensity is controlled by $j_t(L^*L)$, whereas Y_t^W is a noisy observation of $j_t(L + L^*)$. We cannot draw this conclusion, however, as $j_t(L)$ need not commute with A_t or A_t^* , nor with itself at different times.

It is essential, however, that the observation process commutes with itself at different times, and is hence equivalent to a classical stochastic process through the spectral theorem. An observation process that does not obey this property cannot be observed in a single realization of an experiment and is physically meaningless. Let us show that the observations processes we have defined above do obey this property, which is called the *self-nondemolition* property. Let Z be any operator of the form $I \otimes Z_s] \otimes I$ on $\mathcal{H} \otimes \mathcal{F}_s] \otimes \mathcal{F}_s$ and let $t \geq s$. Then the Itô rules give directly

$$U_t^* Z U_t = U_s^* Z U_s + \int_s^t U_\tau^* \mathcal{L}_{L,H}(Z) U_\tau d\tau + \int_s^t U_\tau^* [L^*, Z] U_\tau dA_\tau + \int_s^t U_\tau^* [Z, L] U_\tau dA_\tau^*.$$

Now let $Z = A_s + A_s^*$ or $Z = \Lambda_s$. In both cases $\mathcal{L}_{L,H}(Z) = [Z, L] = 0$ as L and H are system observables and Z is a field observable. Hence $Y_s^W = U_t^* (A_s + A_s^*) U_t$ and $Y_s^\Lambda = U_t^* \Lambda_s U_t$ for all $t \geq s$. It is now easily verified, using the unitarity of U_t and the fact that $A_s + A_s^*$ and Λ_s are commutative processes, that $[Y_t^W, Y_s^W] = [Y_t^\Lambda, Y_s^\Lambda] = 0$ for all t, s . Do note, however, that Y_t^W and Y_t^Λ do not commute with each other; in any experimental realization, we can choose to perform only one of these measurements.

Moving on to the next step in our program, we now wish to use the information gained from the measurement process to infer something about the initial system. To find a least mean square estimate of a system observable X at time t , given the observations Y_t up to this time, we must calculate $\pi_t(X) = \mathbb{P}(j_t(X) | \mathcal{Y}_t)$, where $\mathcal{Y}_t = \text{vN}(Y_s : 0 \leq s \leq t)$ is the (commutative) Von Neumann algebra generated by the observations history. The remainder of these notes is devoted to finding a recursive equation for $\pi_t(X)$ (the *filtering equation*). As we have discussed previously, however, this conditional expectation is only defined if $j_t(X)$ is in the commutant of \mathcal{Y}_t , the intuitive interpretation being that statistical inference of an observable is only physically meaningful if the conditional statistics could possibly be tested through a compatible experiment. Through an entirely identical procedure as in the previous paragraph, we can show that $j_t(X)$ is in the commutant of \mathcal{Y}_t for any system observable X . This is known as the *nondemolition property*.

Remark. Unlike in a classical filtering scenario, we have not added any independent corrupting noise to the observations. Nonetheless, the filtering problem does not trivialize to a problem with complete observations because the system is driven by a quadrature of the field that does not commute with the observations. Hence the problem of partial observations is intrinsic to quantum measurement theory, and does not result only due to technical noise in the detection apparatus. Additional technical noise is however straightforward to take into account as well; we will discuss this possibility in a later section.

As a final note, we remark that we have now obtained a system-theoretic model of our system and observations. For example, the algebra $(\mathcal{B} \otimes \mathcal{W}, \rho \otimes \phi)$ together with the pair

$$\begin{aligned} dj_t(X) &= j_t(\mathcal{L}_{L,H}(X)) dt + j_t([L^*, X]) dA_t + j_t([X, L]) dA_t^* \\ dY_t &= j_t(L + L^*) dt + dA_t + dA_t^* \end{aligned}$$

completely defines a system-observations model, in direct analogy to the system-observation models used throughout classical nonlinear filtering and stochastic control theory.

6 The reference probability method

The goal of this section is to derive the quantum filtering equation of Belavkin, a recursive equation for $\pi_t(X)$, using a method that is close to the classical reference probability method of Zakai [56].

Let us briefly recall the classical procedure of Zakai. In order to simplify the filtering problem, one starts by introducing a new measure, using a Girsanov transformation, under which the observation is a Wiener process. Then various (elementary) properties of the conditional expectation allow the filtering problem to be expressed, and solved, with respect to the new measure. Below we will apply this logic to the quantum filtering problem.

The following filtering problem is considered in this section. Let $Z_t = A_t + A_t^*$ and denote by \mathcal{A}_t the commutant of $\mathcal{C}_t = \text{vN}(Z_s : 0 \leq s \leq t)$. With respect to the state $\mathbb{P} = \rho \otimes \phi$ on the algebra $\mathcal{B} \otimes \mathcal{W}$, we consider the system-observation pair

$$\begin{aligned} dj_t(X) &= j_t(\mathcal{L}_{L,H}(X)) dt + j_t([L^*, X]) dA_t + j_t([X, L]) dA_t^* \\ dY_t &= j_t(L + L^*) dt + dA_t + dA_t^* \end{aligned}$$

where $j_t(X) = U_t^* X U_t$ and U_t is given by (11). We are interested in finding a recursive equation for $\pi_t(X) = \mathbb{P}(j_t(X) | \mathcal{Y}_t)$ where $\mathcal{Y}_t = \text{vN}(Y_s : 0 \leq s \leq t)$.

For future reference we first state two elementary properties of the conditional expectation. Let K_t be an adapted process in \mathcal{A}_t . First, we claim that $\mathbb{P}(K_s | \mathcal{C}_t) = \mathbb{P}(K_s | \mathcal{C}_s)$. This property follows from the fact that $\mathcal{C}_t = \mathcal{C}_s \otimes \mathcal{C}_{[s,t]}$ and that K_s is independent from $\mathcal{C}_{[s,t]}$ by adaptedness. Second, conditional expectations and integrals can be exchanged as follows:

$$\mathbb{P} \left(\int_0^t K_s ds \middle| \mathcal{C}_t \right) = \int_0^t \mathbb{P}(K_s | \mathcal{C}_s) ds, \quad \mathbb{P} \left(\int_0^t K_s dZ_s \middle| \mathcal{C}_t \right) = \int_0^t \mathbb{P}(K_s | \mathcal{C}_s) dZ_s. \quad (13)$$

These properties are immediate if K_t is a simple process, and a proof of the general case is not difficult. Note, however, that these properties would not be as straightforward if we were to replace \mathcal{C}_t by \mathcal{Y}_t . Unlike in the classical Itô theory, quantum Itô theory is grounded in the explicit representation of noise processes on Fock space; a concept such as adaptedness is defined with respect to the Fock space, rather than with respect to the integrator process.

Though representation-free quantum Itô theories have been considered in the literature [1], we choose for simplicity to stick to the Hudson-Parthasarathy theory. In this context, the discussion in the previous paragraph suggests that conditioning is most easily performed if we rotate the problem in such a way that the observations lie entirely in the Fock space. To this end we use another elementary property of conditional expectations: if U is a unitary operator and we define a new state $\mathbb{Q}(X) = \mathbb{P}(U^* X U)$, then $\mathbb{P}(U^* Y U | U^* \mathcal{C} U) = U^* \mathbb{Q}(Y | \mathcal{C}) U$ (this can be verified using the definition of the conditional expectation.) In our setup, this implies that $\pi_t(X) = U_t^* \mathbb{Q}^t(X | \mathcal{C}_t) U_t$ where we have defined the state $\mathbb{Q}^t(X) = \mathbb{P}(U_t^* X U_t)$.

Remark. The time-dependent state \mathbb{Q}^t is precisely the *Schrödinger picture* state; in this picture (used in [18]) the state evolves in time rather than the observables. This approach is ubiquitous in quantum mechanics but is less common in classical stochastics. Here we prefer to work in the Heisenberg picture; we only consider the state \mathbb{Q}^t for fixed time t as an intermediate step in

obtaining the Kallianpur-Striebel formula below, which avoids the technical hurdles due to the explicit representation of quantum noises on the Fock space.

We are now ready to get down to business. Inspired by the classical reference probability method, we seek a change of measure (change of state) that reduces the filtering problem to elementary manipulations. The following lemma describes how conditional expectations are related under a *nondemolition* change of state.

Lemma 6.1: *Let $(\mathcal{N}, \mathbb{P})$ be a noncommutative probability space with normal state \mathbb{P} . Let \mathcal{C} be a commutative von Neumann subalgebra of \mathcal{N} and let \mathcal{A} be its commutant, i.e. $\mathcal{A} = \mathcal{C}' := \{N \in \mathcal{N}; NC = CN, \forall C \in \mathcal{C}\}$. Furthermore, let V be an element in \mathcal{A} such that $\mathbb{P}[V^*V] = 1$. We can define a state on \mathcal{A} by $\mathbb{Q}[A] := \mathbb{P}[V^*AV]$ and we have*

$$\mathbb{Q}[X|\mathcal{C}] = \frac{\mathbb{P}[V^*XV|\mathcal{C}]}{\mathbb{P}[V^*V|\mathcal{C}]}, \quad X \in \mathcal{A}.$$

Proof. Let K be an element of \mathcal{C} . For all $X \in \mathcal{A}$, we can write

$$\begin{aligned} \mathbb{P}[\mathbb{P}[V^*XV|\mathcal{C}]K] &= \mathbb{P}[V^*XKV] = \mathbb{Q}[XK] = \mathbb{Q}[\mathbb{Q}[X|\mathcal{C}]K] = \mathbb{P}[V^*V\mathbb{Q}[X|\mathcal{C}]K] = \\ &= \mathbb{P}[\mathbb{P}[V^*V\mathbb{Q}[X|\mathcal{C}]K|\mathcal{C}]] = \mathbb{P}[\mathbb{P}[V^*V|\mathcal{C}]\mathbb{Q}[X|\mathcal{C}]K], \end{aligned}$$

proving the lemma. □

Remark. It is essential that V is an element of the commutant of \mathcal{C} . The proof would not have worked if V were in \mathcal{N} . The conditional expectation can only be defined from the commutant onto \mathcal{C} and in the first and fourth step of the proof we explicitly used that V is an element of \mathcal{A} .

Our next goal is to find a suitable operator V in order to apply Lemma 6.1 to $\mathbb{Q}^t(X|\mathcal{C}_t)$. Conveniently we have already expressed $\mathbb{Q}^t(X)$ as $\mathbb{P}(U_t^*XU_t)$, and under \mathbb{P} we can directly apply the result Eq. (13). However, U_t is in general not in \mathcal{A}_t , so that Lemma 6.1 can not be applied with $V = U_t$. This is best seen in equation (8), where aside from Z_t the incompatible noise $i(A_t - A_t^*)$ appears as one of the driving terms in the equation for U_t . The problem is resolved by the following technique, which to our knowledge first appeared in a paper by Holevo [34].

Lemma 6.2: *Let V_t be the solution of the QSDE*

$$dV_t = \left\{ L(dA_t^* + dA_t) - \frac{1}{2}L^*Ldt - iHdt \right\} V_t. \quad (14)$$

*Then $V_t \in \mathcal{A}_t$ and $\mathbb{Q}^t(X) = \mathbb{P}(V_t^*XV_t)$.*

Proof. Suppose without loss of generality that $\mathbb{P}(X) = \langle \psi \otimes \Phi, X \psi \otimes \Phi \rangle$ for some vector $\psi \in \mathcal{H}$ (and Φ is the vacuum vector); as \mathcal{H} is finite-dimensional, we can always write any state $\rho \otimes \phi$ as a

convex combination of vector states of this form. Now note that

$$\begin{aligned} d(U_t \psi \otimes \Phi) &= \{LdA_t^* - L^*dA_t - \frac{1}{2}L^*Ldt - iHdt\}U_t \psi \otimes \Phi = \\ &= (LU_t \psi \otimes \Phi_t) \otimes (dA_t^* \Phi_t) - (L^*U_t \psi \otimes \Phi_t) \otimes (dA_t \Phi_t) - \frac{1}{2}L^*LU_t \psi \otimes \Phi dt - iHU_t \psi \otimes \Phi dt = \\ &= (LU_t \psi \otimes \Phi_t) \otimes (dA_t^* \Phi_t) + (LU_t \psi \otimes \Phi_t) \otimes (dA_t \Phi_t) - \frac{1}{2}L^*LU_t \psi \otimes \Phi dt - iHU_t \psi \otimes \Phi dt, \end{aligned}$$

since $dA_t \Phi_t$ is 0. It follows that $U_t \psi \otimes \Phi = V_t \psi \otimes \Phi$, and hence $\mathbb{P}(U_t^* X U_t) = \mathbb{P}(V_t^* X V_t)$ for any $X \in \mathcal{B} \otimes \mathcal{W}$. Moreover, V_t is an element of \mathcal{A}_t ; indeed, equation (14) is driven only by a single commutative noise $A_t + A_t^*$. \square

Putting these results together, we obtain a noncommutative version of the classical Kallianpur-Striebel formula.

Corollary. (Noncommutative Kallianpur-Striebel) Let $\sigma_t(X) = U_t^* \mathbb{P}(V_t^* X V_t | \mathcal{C}_t) U_t$. Then

$$\pi_t(X) = \frac{\sigma_t(X)}{\sigma_t(I)}, \quad \forall X \in \mathcal{B}. \quad (15)$$

We now obtain an explicit expression for $\sigma_t(X)$. Using the quantum Itô rules

$$V_t^* X V_t = X + \int_0^t V_s^* \mathcal{L}_{L,H}(X) V_s ds + \int_0^t V_s^* (L^* X + X L) V_s d(A_s + A_s^*).$$

Using Eq. (13) we obtain

$$\mathbb{P}[V_t^* X V_t | \mathcal{C}_t] = \mathbb{P}[X] + \int_0^t \mathbb{P}[V_s^* \mathcal{L}_{L,H}(X) V_s | \mathcal{C}_s] ds + \int_0^t \mathbb{P}[V_s^* (L^* X + X L) V_s | \mathcal{C}_s] d(A_s + A_s^*).$$

Another application of the quantum Itô rules now yields immediately the *Belavkin-Zakai equation*

$$d\sigma_t(X) = \sigma_t(\mathcal{L}_{L,H}(X))dt + \sigma_t(L^* X + X L)dY_t. \quad (16)$$

Remark. It follows from the linearity and complete positivity of the conditional expectation that σ_t is a linear positive functional on \mathcal{B} , i.e. apart from the normalisation it is a (random) state. Hence Eq. (16) is a noncommutative analogue of the Zakai equation in classical filtering theory and propagates the unnormalized conditional state σ_t .

By applying the (quantum) Itô rules to the Kallianpur-Striebel formula, we obtain an expression for the normalized conditional state

$$d\pi_t(X) = \pi_t(\mathcal{L}_{L,H}(X))dt + \left(\pi_t(L^* X + X L) - \pi_t(L^* + L)\pi_t(X) \right) \left(dY_t - \pi_t(L^* + L)dt \right). \quad (17)$$

This *Belavkin-Kushner-Stratonovich quantum filtering equation* is a quantum analogue of the classical Kushner-Stratonovich equation.

We conclude this section with an investigation of the *innovations process* $d\bar{Z}_t = dY_t - \pi_t(L^* + L)dt$. The general theorem below [13, 18] shows that \bar{Z}_t is a martingale; but as $d\bar{Z}_t^2 = dt$ it must be

a Wiener process by Lévy's theorem (the latter can be applied directly if we use the spectral theorem to find a classical representation of \overline{Z}_t .) Also this property is in complete analogy with the innovations process in classical filtering theory.

Theorem 6.3: Let $dZ_t = a_t d\Lambda_t + b_t dA_t^* + b_t^* dA_t$ and $Z_0 = 0$ where $a_t \in \mathbb{R}$ and $b_t \in \mathbb{C}$ for all $t \geq 0$. Define the innovating martingale \overline{Z}_t by

$$\overline{Z}_t := U_t^* Z_t U_t - \left(\int_0^t a_s \pi_s(L^* L) + b_s \pi_s(L^*) + b_s^* \pi_s(L) ds \right).$$

Then \overline{Z}_t is a martingale, i.e. for all $t \geq s \geq 0$ we have $\mathbb{P}[\overline{Z}_t | \mathcal{Y}_s] = \overline{Z}_s$.

Proof. We need to prove that $\mathbb{P}[\overline{Z}_t - \overline{Z}_s | \mathcal{Y}_s] = 0$ for all $t \geq s \geq 0$. This means we have to prove for all $t \geq s \geq 0$ and $K \in \mathcal{Y}_s$ that $\mathbb{P}[(\overline{Z}_t - \overline{Z}_s)K] = 0$. This is equivalent to

$$\mathbb{P}(U_t^* Z_t U_t K) - \mathbb{P}(U_s^* Z_s U_s K) = \int_s^t \mathbb{P} \left(a_r \pi_r(L^* L) K + b_r \pi_r(L^*) K + b_r^* \pi_r(L) K \right) ds,$$

for all $t \geq s \geq 0$ and $K \in \mathcal{Y}_s$. Since $K \in \mathcal{Y}_s$ it can be written as $K = U_s^* C U_s = U_t^* C U_t$ for some $C \in \mathcal{C}_s$. Furthermore, if C runs through \mathcal{C}_s , K runs through \mathcal{Y}_s . For $t = s$ the above equation is true, therefore it remains to show that for all $C \in \mathcal{C}_s$

$$\begin{aligned} d\mathbb{P}(U_t^* Z_t C U_t) &= \mathbb{P}(a_r \pi_r(L^* L C) + b_r \pi_r(L^* C) + b_r^* \pi_r(L C)) dt \\ &= \mathbb{P}(U_t^* (a_r L^* L C + b_r L^* C + \overline{b}_r L C) U_t) dt. \end{aligned}$$

But this is just an exercise in applying the quantum Itô rules. □

7 More examples

7.1 Controlled quantum diffusions

Up to this point we have considered filtering only for simple quantum systems interacting with a field. However, an important application of quantum filtering theory is in *quantum control*—we can feed back a signal to the initial system based on the observations we have made, in real time, in order to achieve some particular control goal. In control theory, the filter is often used as an intermediate step in generating the control signal. Before we can do this, however, we must show that the filtering equations retain their form even in the presence of feedback. This can be done in a natural way in the reference probability approach.

The basic object we need is a *controlled quantum diffusion*. As before, we fix the algebra $\mathcal{B} \otimes \mathcal{W}$. The controlled diffusion consists of two things:

- The output noise Z_t (we will choose $Z_t = A_t + A_t^*$).

- The controlled Hudson-Parthasarathy QSDE¹

$$dU_t = \left\{ L_t dA_t^* - L_t^* dA_t - \frac{1}{2} L_t^* L_t dt - iH_t dt \right\} U_t$$

where for all t , H_t and L_t are affiliated to $\mathcal{B} \otimes \mathcal{C}_t$ ($\mathcal{C}_t = \vee N(Z_s : 0 \leq s \leq t)$), and H_t is self-adjoint.

As before, we find that this model gives rise to the system-observation pair

$$\begin{aligned} dj_t(X) &= j_t(\mathcal{L}_{L_t, H_t}(X)) dt + j_t([L_t^*, X]) dA_t + j_t([X, L_t]) dA_t^* \\ dY_t &= j_t(L_t + L_t^*) dt + dA_t + dA_t^* \end{aligned}$$

where $j_t(X) = U_t^* X U_t$ is the flow and $Y_t = U_t^* Z_t U_t$ is the output process.

To expose the connection with control, consider the simplest case where $L_t = L \in \mathcal{B}$ is fixed and $H_t = H \otimes u_t(Z_{s \leq t})$, i.e. H is a fixed Hamiltonian in \mathcal{B} and u_t is some real function of the history of the output noise up to time t . Then we get

$$\begin{aligned} dj_t(X) &= u_t(Y_{s \leq t}) j_t(i[H, X]) dt + j_t(\mathcal{L}_{L, I}(X)) dt + j_t([L^*, X]) dA_t + j_t([X, L]) dA_t^* \\ dY_t &= j_t(L + L^*) dt + dA_t + dA_t^* \end{aligned}$$

where it is important to note that by pulling it outside $j_t(\cdot)$, u_t becomes a function of the output process! This is just the model we considered before, with the difference that here the Hamiltonian term is modulated by a control signal which is some (arbitrary) function of the observations history. This is precisely the situation in (Hamiltonian) feedback control, the idea being that we choose the function u_t so that a particular control goal is achieved. The general model of a controlled quantum diffusion allows us to make both H and L arbitrary functions of the observations history, and thus provides a general model for quantum systems with feedback.

The question we ask here is, what is the form of the filter when we allow for feedback? It turns out that nothing much changes in the derivation of the filter using the reference probability method. We can follow all the same steps to obtain the Kallianpur-Striebel formula

$$\pi_t(X) = \mathbb{P}(j_t(X) | \vee N(Y_s : 0 \leq s \leq t)) = \frac{\sigma_t(X)}{\sigma_t(I)}, \quad \sigma_t(X) = U_t^* \mathbb{P}(V_t^* X V_t | \mathcal{C}_t) U_t$$

with the \mathcal{C}'_t -affiliated change of measure

$$dV_t = \left\{ L_t dZ_t - \frac{1}{2} L_t^* L_t dt - iH_t dt \right\} V_t.$$

By applying the quantum Itô rules, we now obtain the Zakai equation

$$d\sigma_t(X) = \sigma_t(\mathcal{L}_{L_t, H_t}(X)) dt + \sigma_t(L_t^* X + X L_t) dY_t.$$

In particular, suppose that as before $L_t = L$, $H_t = H \otimes u_t(Z_{s \leq t})$. Note that $\mathbb{P}(V_t^* i[H_t, X] V_t | \mathcal{C}_t) = u_t(Z_{s \leq t}) \mathbb{P}(V_t^* i[H, X] V_t | \mathcal{C}_t)$ as V_t commutes with $Z_{s \leq t}$ and by the module property of the conditional expectation. Thus in this case

$$d\sigma_t(X) = u_t(Y_{s \leq t}) \sigma_t(i[H, X]) dt + \sigma_t(\mathcal{L}_{L, I}(X)) dt + \sigma_t(L^* X + X L) dY_t.$$

¹For existence and uniqueness, see e.g. [30].

We see that as expected, the filter for a quantum diffusion with feedback has precisely the same form as the filter obtained previously with feedback added in a naive way. This provides a foundation for the use of the filter in quantum feedback control.

7.2 Imperfect observations

We consider the same filtering problem as in the section 6, except that now we add additional corrupting noise to the observation

$$dY_t = j_t(L + L^*) dt + dA_t + dA_t^* + \kappa(dB_t + dB_t^*).$$

Here $\kappa \geq 0$ and B_t is an additional field, the *corrupting noise*, which is independent of A_t ; i.e., we have tensored another copy of the Fock space \mathcal{F} onto the Hilbert space, and B_t is the fundamental process on this space. We now take $Z_t = A_t + A_t^* + \kappa(B_t + B_t^*)$; as before $\mathcal{C}_t = \text{vN}(Z_s : 0 \leq s \leq t)$ and \mathcal{A}_t is its commutant. Note that we still have $Y_t = U_t^* Z_t U_t$, etc.

Up to and including the Kallianpur-Striebel formula, this case is identical to the one treated in section 6. As before, we have

$$V_t^* X V_t = X + \int_0^t V_s^* \mathcal{L}_{L,H}(X) V_s ds + \int_0^t V_s^* (L^* X + X L) V_s d(A_s + A_s^*).$$

However, we cannot directly apply Eq. (13) at this point, as in this case the stochastic integral with respect to $A_s + A_s^*$ does not correspond to the integral with respect to Z_s . Let us write

$$A_t + A_t^* = \alpha Z_t + [(1 - \alpha)(A_t + A_t^*) - \alpha\kappa(B_t + B_t^*)],$$

which holds for any $\alpha \in \mathbb{R}$. Now note that if we choose the particular value $\alpha = (1 + \kappa^2)^{-1}$, then the noise Z_t is independent from $M_t = (1 - \alpha)(A_t + A_t^*) - \alpha\kappa(B_t + B_t^*)$. By an elementary argument, it follows that

$$\mathbb{P} \left(\int_0^t K_s dM_s \middle| \mathcal{C}_t \right) = 0.$$

This, together with Eq. (13), directly gives the Belavkin-Zakai equation with imperfect observations

$$d\sigma_t(X) = \sigma_t(\mathcal{L}_{L,H}(X))dt + (1 + \kappa^2)^{-1} \sigma_t(L^* X + X L) dY_t. \quad (18)$$

7.3 Photon counting observations

Once again we consider the same system; instead of homodyne detection, however, we now perform photon counting in the field: i.e., $Y_t = U_t^* \Lambda_t U_t$ which gives

$$dY_t = d\Lambda_t + j_t(L) dA_t^* + j_t(L^*) dA_t + j_t(L^* L) dt.$$

We would like to follow the same procedure as for homodyne detection. The following lemma, which replaces Lemma 6.2, suggests how to proceed. The proof is identical to that of Lemma 6.2.

Lemma 7.1: Let U'_t be the solution of the Hudson-Parthasarathy equation

$$dU'_t = \left\{ L' dA_t^* - L'^* dA_t - \frac{1}{2} L'^* L' dt - iH' dt \right\} U'_t$$

and let V'_t be the solution of

$$dV'_t = \left\{ L'(d\Lambda_t + dA_t^* + dA_t + dt) - \frac{1}{2} L'^* L' dt - L' dt - iH' dt \right\} V'_t.$$

Then $V_t \in \text{vN}(\Lambda_s + A_s^* + A_s + s : 0 \leq s \leq t)$ and $\mathbb{P}(U_t'^* X U'_t) = \mathbb{P}(V_t'^* X V'_t)$.

Define $Z_t = \Lambda_t + A_t^* + A_t + t$, and as before \mathcal{A}_t is the commutant of $\mathcal{C}_t = \text{vN}(Z_s : 0 \leq s \leq t)$ and $\mathcal{Y}_t = \text{vN}(Y_s : 0 \leq s \leq t)$. Lemma 7.1 directly provides us with a nondemolition change of measure, provided that we rotate our problem so that $\mathcal{Y}_t = U_t'^* \mathcal{C}_t U'_t$ using a suitable unitary U'_t . Then, defining $\sigma_t(X) = U_t'^* \mathbb{P}(V_t'^* X V'_t | \mathcal{C}_t) U'_t$, the Kallianpur-Striebel formula holds for $\sigma_t(X)$.

Define W_t as the solution of the QSDE

$$dW_t = [dA_t - dA_t^* - \frac{1}{2} dt] W_t$$

(compare with Eq. (6).) Using the quantum Itô rules one can verify that $\Lambda_t = W_t^* Z_t W_t$. But recall that $Y_t = U_t^* \Lambda_t U_t = U_t^* W_t^* Z_t W_t U_t$; thus $U'_t = W_t U_t$ is our rotation of choice. Another application of the quantum Itô rules gives

$$dU'_t = \left\{ (L - 1) dA_t^* - (L^* - 1) dA_t - \frac{1}{2} (L^* L + I - 2L + 2iH) dt \right\} U'_t,$$

which corresponds to the nondemolition change of measure

$$dV'_t = \left\{ (L - 1) dZ_t - \frac{1}{2} (L^* L - I + 2iH) dt \right\} V'_t.$$

For $X \in \mathcal{B}$, we obtain using the quantum Itô rules

$$dV_t'^* X V'_t = V_t'^* (\mathcal{L}_{L,H}(X)) V'_t dt + V_t'^* (L^* X L - X) V'_t (dZ_t - dt).$$

Finally we obtain using the definition of σ_t , Eq. (13), and the quantum Itô rules

$$d\sigma_t(X) = \sigma_t(\mathcal{L}_{L,H}(X)) dt + (\sigma_t(L^* X L) - \sigma_t(X)) (dY_t - dt). \quad (19)$$

which is the Belavkin-Zakai equation for counting observations.

Using the Kallianpur-Striebel formula we can now obtain an expression for the normalized conditional state

$$d\pi_t(X) = \pi_t(\mathcal{L}_{L,H}(X)) dt + \left(\frac{\pi_t(L^* X L)}{\pi_t(L^* L)} - \pi_t(X) \right) (dY_t - \pi_t(L^* L) dt),$$

which is the quantum filtering equation for photon counting. By Theorem 6.3, we see that the innovations process $d\bar{Z}_t = dY_t - \pi_t(L^* L) dt$ is a martingale. Note that this implies that in terms of the conditional state, \bar{Z}_t is a counting process with rate $\pi_t(L^* L)$.

7.4 Other filters

There is a wide array of other relevant examples, which we do not discuss here. For risk-sensitive filtering equations and their use in risk-sensitive control we refer to [36, 37]. Moreover, sometimes the input noise is not taken to be in the vacuum state, as is the case for squeezed or thermal noise. For a general treatment we refer to [12], and for the specific case of squeezed noise see [16].

Acknowledgment

This work was supported by the ARO under Grant DAAD19-03-1-0073.

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